

PROPERTIES OF THE SEQUENCE $\{Z[t_\nu(\tau)]\}$, JACOB'S LADDERS AND NEW KIND OF INFINITE SET OF METAMORPHOSIS OF MAIN MULTIFORM

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ABSTRACT. In this paper we study properties of some sums of members of the sequence $\{Z[t_\nu(\tau)]\}$. Our results are expressed in statements proving essential influence of the Lindelöf hypothesis on corresponding formulae. In this paper: the parts 1 – 6 are English version of our paper [6], and the part 7 of this work contains current results, namely new set of metamorphosis of the main multiform from our paper [7].

1. INTRODUCTION

1.1. Let us remind the Riemann-Siegel formula

$$(1.1) \quad \begin{aligned} Z(t) &= 2 \sum_{n \leq \rho(t)} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + \mathcal{O}(t^{-1/4}), \\ \vartheta(t) &= -\frac{t}{2} \ln \pi + \operatorname{Im} \ln \Gamma\left(\frac{1}{4} + i\frac{t}{2}\right), \quad \rho(t) = \sqrt{\frac{t}{2\pi}}, \end{aligned}$$

(see [8], pp. 79, 329). In the paper [5] we have defined the following two notions in connection with the formula (1.1):

(a) the class of sequences

$$\{t_\nu(\tau)\}$$

by the condition

$$(1.2) \quad \begin{aligned} \vartheta[t_\nu(\tau)] &= \pi\nu + \tau, \quad \tau \in [-\pi, \pi], \quad t(0) = t_\nu, \\ \nu_0 &\leq \nu, \quad \nu_0 \in \mathbb{N}, \end{aligned}$$

where ν_0 is sufficiently big and fixed number,

(b) the following classes of disconnected sets

$$(1.3) \quad \begin{aligned} \mathbb{G}_1 &= \mathbb{G}_1(x, T, H) = \\ &= \bigcup_{T \leq t_{2\nu} \leq T+H} \{t : t_{2\nu}(-x) < t < t_{2\nu}(x)\}, \quad x \in (0, \pi/2], \\ \mathbb{G}_2 &= \mathbb{G}_1(y, T, H) = \\ &= \bigcup_{T \leq t_{2\nu+1} \leq T+H} \{t : t_{2\nu+1}(-y) < t < t_{2\nu+1}(y)\}, \quad y \in (0, \pi/2], \\ H &\in (0, H_1], \quad H_1 = T^{1/6+\epsilon}, \end{aligned}$$

where ϵ is positive and arbitrarily small number.

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Further, by making use of a synthesis of discrete and continuous method, that is (see (1.2)):

- (c) summation according to ν ,
- (d) integration according to τ ,

we have obtained the following new mean-value formulae (see [5])

$$(1.4) \quad \begin{aligned} \frac{1}{m\{\mathbb{G}_1\}} \int_{\mathbb{G}_1} Z(t) dt &\sim 2 \frac{\sin x}{x}, \\ \frac{1}{m\{\mathbb{G}_2\}} \int_{\mathbb{G}_2} Z(t) dt &\sim -2 \frac{\sin y}{y}, \\ T &\rightarrow \infty, \end{aligned}$$

on discrete sets $\mathbb{G}_1, \mathbb{G}_2$, where $m\{\cdot\}$ stands for the measure of the corresponding set.

Remark 1. Formulae (1.4) are the first mean-value theorems linear in the function $Z(t)$ in the theory of the Riemann zeta-function.

1.2. In this paper we shall study the following sums

$$(1.5) \quad \begin{aligned} &\sum_{T \leq t_\nu \leq T+H} \{Z[t_\nu(\tau)] - Z(t_\nu)\}, \\ &\sum_{T \leq t_\nu \leq T+H} (-1)^\nu \{Z[t_\nu(\tau)] - Z(t_\nu)\}, \\ &H \in (0, T^{1/6+\epsilon}], \end{aligned}$$

and corresponding integral formulae also.

Remark 2. The Lindelöf hypothesis gives an essential influence on resulting asymptotic formulae.

1.3. It is the most astonishing fact for the author that the expression

$$(1.6) \quad Z[t_\nu(\tau)] - Z(t_\nu)$$

coming from our old (1983) paper [6] is close connected with the ζ -factorization and with the metamorphosis of main multiform from our last (2015) paper [7].

Remark 3. Namely, from (1.6) it follows that there is an essentially new set of metamorphosis of the main multiform (see [7], (2.4)).

2. THEOREM 1

Let

$$(2.1) \quad F(\tau, T, H) = \sum_{T \leq t_\nu \leq T+H} Z[t_\nu(\tau)]$$

and

$$(2.2) \quad S(a, b) = \sum_{1 \leq a \leq n < b \leq 2a} n^{it}, \quad b \leq \sqrt{\frac{t}{2\pi}}$$

stands for elementary trigonometric sum. The following theorem holds true.

Theorem 1. If

$$(2.3) \quad |S(a, b)| < A(\Delta)\sqrt{at}^\Delta, \quad \Delta \in (0, 1/6],$$

(see (2.2)), then

$$(2.4) \quad \begin{aligned} F(\tau, T, H) - F(0, T, H) &= \mathcal{O}(T^\Delta \ln T), \\ H &\in (0, T^{1/6+\epsilon}] \end{aligned}$$

uniformly for $\tau \in [-\pi, \pi]$.

In the case $\Delta = 1/6$ we have the following

Corollary 1.

$$F(\tau, T, H) - F(0, T, H) = \mathcal{O}(T^{1/6+\epsilon})$$

uniformly for $\tau \in [-\pi, \pi]$.

If the Lindelöf hypothesis holds true, then

$$\Delta = \frac{\epsilon}{2},$$

(comp. [1], p. 89), and we have the following

Corollary 2. On Lindelöf hypothesis

$$F(\tau, T, H) - F(0, T, H) = \mathcal{O}(T^\epsilon)$$

uniformly for $\tau \in [-\pi, \pi]$.

3. THEOREM 2

First of all the following formula holds true

$$(3.1) \quad \sum_{T \leq t_\nu \leq T+H} (-1)^\nu Z(t_\nu) = \frac{1}{\pi} H \ln \frac{T}{2\pi} + \mathcal{O}(T^\Delta \ln T)$$

in the case (2.3) (see [3], p. 89, (5)). Next, we have proved (see [5], (5.1)) that in the case (2.3) the formula

$$(3.2) \quad \begin{aligned} \sum_{T \leq t_\nu \leq T+H} (-1)^\nu Z[t_\nu(\tau)] &= \\ &= \frac{1}{\pi} H \ln \frac{T}{2\pi} \cos \tau + \mathcal{O}(T^\Delta \ln T) \end{aligned}$$

follows, where the \mathcal{O} -estimate is true uniformly for $\tau \in [-\pi, \pi]$.

Further, from (3.1), (3.2) the formula

$$(3.3) \quad \begin{aligned} \sum_{T \leq t_\nu \leq T+H} (-1)^\nu \{Z[t_\nu(\tau)] - Z(t_\nu)\} &= \\ &= -\frac{2}{\pi} H \ln \frac{T}{2\pi} \sin^2 \frac{\tau}{2} + \mathcal{O}(T^\Delta \ln T) \end{aligned}$$

follows. Consequently, we have by (2.4), (3.3) the following

Theorem 2. It follows from (2.3) that

$$\begin{aligned}
 (3.4) \quad & \sum_{T \leq t_{2\nu} \leq T+H} (-1)^\nu \{Z[t_{2\nu}(\tau)] - Z(t_{2\nu})\} = \\
 & = -\frac{1}{\pi} H \ln \frac{T}{2\pi} \sin^2 \frac{\tau}{2} + \mathcal{O}(T^\Delta \ln T), \\
 & \sum_{T \leq t_{2\nu+1} \leq T+H} (-1)^\nu \{Z[t_{2\nu+1}(\tau)] - Z(t_{2\nu+1})\} = \\
 & = \frac{1}{\pi} H \ln \frac{T}{2\pi} \sin^2 \frac{\tau}{2} + \mathcal{O}(T^\Delta \ln T),
 \end{aligned}$$

where the \mathcal{O} -estimates hold true uniformly for $\tau \in [-\pi, \pi]$.

Remark 4. Our formulae (3.3), (3.4) are asymptotic ones in the case

$$(3.5) \quad H = T^\Delta \ln T.$$

4. THEOREM 3

Of course,

$$\begin{aligned}
 (4.1) \quad & \frac{1}{t_{2\nu}(x) - t_{2\nu}(-x)} \int_{t_{2\nu}(-x)}^{t_{2\nu}(x)} Z(t) dt = Z[\xi_{2\nu}(x)], \\
 & \frac{1}{t_{2\nu+1}(y) - t_{2\nu+1}(-y)} \int_{t_{2\nu+1}(-y)}^{t_{2\nu+1}(y)} Z(t) dt = Z[\xi_{2\nu+1}(y)],
 \end{aligned}$$

where

$$\begin{aligned}
 \xi_{2\nu}(x) & \in (t_{2\nu}(-x), t_{2\nu}(x)), \\
 \xi_{2\nu+1}(y) & \in (t_{2\nu+1}(-y), t_{2\nu+1}(y)),
 \end{aligned}$$

and the numbers

$$Z[\xi_{2\nu}(x)], \quad Z[\xi_{2\nu+1}(y)]$$

are the mean-values of the function $Z(t)$ with respect to corresponding segments. Also, in this direction, there are asymptotic formulae for the sums of differences

$$\begin{aligned}
 (4.2) \quad & Z[\xi_{2\nu}(x)] - Z(t_{2\nu}), \\
 & Z[\xi_{2\nu+1}(y)] - Z(t_{2\nu+1}).
 \end{aligned}$$

Remark 5. The behavior of the differences (4.2) is very irregular.

However, the following theorem holds true.

Theorem 3. It follows from (2.3) that

$$\begin{aligned}
 (4.3) \quad & \sum_{T \leq t_{2\nu} \leq T+H} \{Z[\xi_{2\nu}(x)] - Z(t_{2\nu})\} = \\
 & = -\frac{1}{2\pi} \left(1 - \frac{\sin x}{x}\right) H \ln \frac{T}{2\pi} + \mathcal{O}(T^\Delta \ln T), \\
 & \sum_{T \leq t_{2\nu+1} \leq T+H} \{Z[\xi_{2\nu+1}(y)] - Z(t_{2\nu+1})\} = \\
 & = \frac{1}{2\pi} \left(1 - \frac{\sin y}{y}\right) H \ln \frac{T}{2\pi} + \mathcal{O}(T^\Delta \ln T).
 \end{aligned}$$

5. PROOF OF THEOREM 1

First of all, we obtain from (1.1) (see [2], (57)) that

$$Z(t) = 2 \sum_{n < P_0} \frac{1}{\sqrt{n}} \cos\{\vartheta(t) - t \ln n\} + \mathcal{O}(t^{-1/4}),$$

$$t \in [T, T + H], \quad H \in (0, \sqrt[4]{T}], \quad P_0 = \sqrt{\frac{T}{2\pi}}.$$

Since (see (1.2))

$$\vartheta[t_\nu(\tau)] - \vartheta(t_\nu) = \tau,$$

then (see [2], (41), (42))

$$(5.1) \quad t_\nu(\tau) - t_\nu = \frac{\tau}{\ln P_0} + \mathcal{O}\left(\frac{H}{T \ln^2 T}\right), \quad t_\nu \in [T, T + H].$$

Next,

$$\begin{aligned} \sin\left(\frac{\pi}{2} - \frac{t_\nu(\tau) - t_\nu}{2} \ln n\right) &= \sin\left\{\frac{\tau}{\pi} X(n)\right\} + \mathcal{O}\left(\frac{H}{T \ln T}\right), \\ \sin\left(\frac{\pi}{2} - \frac{t_\nu(\tau) + t_\nu}{2} \ln n\right) &= \sin\left\{\frac{\tau}{\pi} X(n) - t_\nu \ln n\right\} + \mathcal{O}\left(\frac{H}{T \ln T}\right), \\ X(n) &= \frac{\pi}{2 \ln P_0} \ln \frac{P_0}{n}, \\ 0 < X(n) &\leq \frac{\pi}{2}, \quad 1 \leq n < P_0. \end{aligned}$$

Now (see (1.2))

$$\begin{aligned} Z[t_\nu(\tau)] - Z(t_\nu) &= \\ &= 4(-1)^{\nu+1} \sum_{n < P_0} \frac{1}{\sqrt{n}} \sin\left\{\frac{\tau}{\pi} X(n)\right\} \sin\left\{\frac{\tau}{\pi} X(n) - t_\nu \ln n\right\} + \\ &+ \mathcal{O}\left(\sqrt[4]{T} \frac{H}{T \ln T}\right) + \mathcal{O}(T^{-1/4}) = \\ (5.2) \quad &= 4(-1)^{\nu+1} \sum_{n < P_0} \frac{1}{\sqrt{n}} \sin^2\left\{\frac{\tau}{\pi} X(n)\right\} \cos\{t_\nu \ln n\} + \\ &+ 2(-1)^\nu \sum_{n < P_0} \frac{1}{\sqrt{n}} \sin\left\{\frac{2\tau}{\pi} X(n)\right\} \sin\{t_\nu \ln n\} + \\ &+ \mathcal{O}(T^{-1/4}). \end{aligned}$$

Hence (see (2.1), [2], (59))

$$\begin{aligned} F(\tau, T, H) - F(0, T, H) &= \sum_{T \leq t_\nu \leq T+H} \{Z[t_\nu(\tau)] - Z(t_\nu)\} = \\ &= -4 \sum_{n < P_0} \frac{1}{\sqrt{n}} \sin^2\left\{\frac{\tau}{\pi} X(n)\right\} \cdot \sum_{T \leq t_\nu \leq T+H} (-1)^\nu \cos\{t_\nu \ln n\} + \\ &+ 2 \sum_{n < P_0} \sin\left\{\frac{\tau}{\pi} X(n)\right\} \cdot \sum_{T \leq t_\nu \leq T+H} (-1)^\nu \sin\{t_\nu \ln n\} + \\ &+ \mathcal{O}(\ln T) = \\ &= -4w_1 + 2w_2 + \mathcal{O}(\ln T). \end{aligned}$$

The sum w_1 contains the following typical member

$$w_{11} = \sum_{n < P_0} \sin^2 \left\{ \frac{\tau}{\pi} X(n) \right\} \frac{\tan \frac{\omega}{2}}{\sqrt{n}} \sin \varphi,$$

(comp. [2], (54)), where

$$\begin{aligned} \varphi &= t_\nu \ln n, \\ \frac{\omega}{2} &= \frac{\pi}{2} \frac{\ln n}{\ln P_0} = \frac{\pi}{2} - X(n), \\ \tan \frac{\omega}{2} &= \cot X(n), \end{aligned}$$

(comp. [2], (43), (50)). Consequently,,

$$(5.3) \quad w_{11} = \left(\frac{\tau}{\pi} \right)^2 \sum_{n < P_0} X \frac{\sin^2 \left(\frac{\tau}{\pi} X \right)}{\left(\frac{\tau}{\pi} X \right)^2} \frac{X}{\sin X} \cos X \frac{1}{\sqrt{n}} \sin \varphi.$$

Since in the case (2.3) we have that

$$\sum_{1 \leq n < P_1 \leq P_0} \frac{1}{\sqrt{n}} \sin \varphi = \mathcal{O}(T^\Delta \ln T)$$

then, by making use the Abel's transformation several-time in (5.3), we obtain the estimate

$$w_{11} = \mathcal{O}(T^\Delta \ln T),$$

and, consequently,

$$w_1 = \mathcal{O}(T^\Delta \ln T)$$

uniformly for $\tau \in [-\pi, \pi]$. Because of the simple identity

$$\sin(t_\nu \ln n) = \cos(t_\nu \ln n - \pi/2)$$

we obtain by a similar way the estimate for w_2 too.

6. PROOF OF THEOREM 3

Since (see [5], (2.1))

$$t_{2\nu}(x) - t_{2\nu}(-x) = \frac{2x}{\ln P_0} + \mathcal{O}\left(\frac{xH}{T \ln^2 T}\right),$$

and by (2.3)

$$Z(t) = \mathcal{O}(T^\Delta \ln T),$$

then (see (2.3), (4.1), [5], (6.2))

$$\begin{aligned} \int_{-x}^x Z[t_{2\nu}(\tau)] d\tau &= \ln P_0 \int_{t_{2\nu}(-x)}^{t_{2\nu}(x)} Z(t) dt + \mathcal{O}(xHT^{-5/6}) = \\ &= [t_{2\nu}(x) - t_{2\nu}(-x)] Z[\xi_{2\nu}(x)] \ln P_0 + \mathcal{O}(xHT^{-5/6}) = \\ &= 2xZ[\xi_{2\nu}(x)] + \mathcal{O}(xHT^{-5/6}). \end{aligned}$$

Hence,

$$\begin{aligned} & \int_{-x}^x \sum_{T \leq t_{2\nu} \leq T+H} \{Z[t_{2\nu}(\tau)] - Z(t_{2\nu})\} d\tau = \\ & = 2x \sum_{T \leq t_{2\nu} \leq T+H} \{Z[\xi_{2\nu}(x)] - Z(t_{2\nu})\} + \\ & + \mathcal{O}(xHT^{-5/6}H \ln T), \end{aligned}$$

where we have used the formula (see [2], (2.3))

$$\sum_{T \leq t_{2\nu} \leq T+H} 1 \sim \frac{1}{2\pi} H \ln \frac{T}{2\pi}.$$

Consequently, the integration of the first formula in (3.4) by $\tau \in [-\pi, \pi]$ together with the formula

$$\int_{-x}^x \sin^2 \frac{\tau}{2} d\tau = x - \sin x$$

gives the first formula in (4.3), and, by a similar way, we obtain also the second formula in (4.3).

7. ON NEW KIND OF METAMORPHOSIS OF MAIN MULTIFORM

7.1. First of all, we define the following sets

$$(7.1) \quad \begin{aligned} w_\nu(T, H) &= \{\tau : \tau \in (0, \pi], Z[t_\nu(\tau)] \neq Z(t_\nu)\}, \\ t_\nu &\in [T, T+H], H \in \left[\frac{\ln \ln T}{\ln T}, T^{\frac{1}{\ln \ln T}} \right]. \end{aligned}$$

The lower bound for H in (7.1) is chosen with respect to the formula (see (5.1))

$$(7.2) \quad t_\nu(\tau) - t_\nu \sim \frac{\tau}{\ln P_0}, \quad T \rightarrow \infty.$$

Next, by the Newton-Leibniz formula

$$(7.3) \quad \left| \int_{t_\nu}^{t_\nu(\tau)} Z'(u) du \right| = |Z[t_\nu(\tau)] - Z(t_\nu)|, \quad \tau \in w_\nu.$$

Remark 6. It is a new thing that the elementary identity (7.3) has a nontrivial continuation by means of Jacob's ladders and reversely iterated integrals.

7.2. Since (see (7.2))

$$t_\nu(\tau) - t_\nu = o\left(\frac{T}{\ln T}\right),$$

then we obtain from (7.3) by [7], (5.2), (5.3) that

$$(7.4) \quad \begin{aligned} & \prod_{r=0}^{k-1} |Z[\varphi_1^r(\beta)]| \sim \\ & \sim \sqrt{\frac{|Z[t_\nu(\tau)] - Z[t_\nu]|}{\widehat{t_\nu(\tau)} - \widehat{t_\nu}^k}} \ln^k T \frac{1}{\sqrt{|Z'[\varphi_1^k(\beta)]|}}, \quad \tau \in w_\nu. \end{aligned}$$

Next, we obtain from (7.4) by [7], (5.4) – (5.7) that

$$\begin{aligned} \prod_{l=1}^k |Z[\bar{\alpha}_l]| &\sim \frac{\Lambda_1}{\sqrt{|Z'(\bar{\alpha}_0)|}}, \quad \tau \in w_\nu, \\ \bar{\alpha}_r &= \bar{\alpha}_r(T, H, k, \nu, \tau), \quad r = 0, 1, \dots, k, \\ \bar{\alpha}_r &\neq \gamma: \quad \zeta\left(\frac{1}{2} + i\gamma\right) = 0, \\ \Lambda_1 &= \Lambda_1(T, H, k, \nu, \tau) = \sqrt{\frac{|Z[t_\nu(\tau)] - Z[t_\nu]|}{\widehat{t_\nu(\tau)}^k - \widehat{t_\nu}^k}} \ln^k T. \end{aligned}$$

Consequently, the following theorem holds true (comp. [7], Theorem)

Theorem 4. Let

$$k = 1, \dots, k_0, \quad k_0 \in \mathbb{N}$$

for every fixed k_0 and let

$$H = H(T) \in \left[\frac{\ln \ln T}{\ln T}, T^{\frac{1}{\ln \ln T}} \right]$$

for every sufficiently big $T > 0$. Then for every k, H and for every $\tau \in w_\nu$ there are functions

$$\begin{aligned} \bar{\alpha}_r &= \bar{\alpha}_r(T, H, k, \nu, \tau) > 0, \quad r = 0, 1, \dots, k, \\ \bar{\alpha}_r &\neq \gamma: \quad \zeta\left(\frac{1}{2} + i\gamma\right) = 0, \end{aligned}$$

such that the following ζ -factorization formula

$$\begin{aligned} \frac{\Lambda_1}{\sqrt{|Z'(\bar{\alpha}_0)|}} &\sim \prod_{l=1}^k |Z(\bar{\alpha}_l)|, \quad \tau \in w_\nu, \\ \Lambda_1 &= \Lambda_1(T, H, k, \nu, \tau) = \sqrt{\frac{|Z[t_\nu(\tau)] - Z[t_\nu]|}{\widehat{t_\nu(\tau)}^k - \widehat{t_\nu}^k}} \ln^k T \end{aligned}$$

holds true. Moreover, the sequence

$$\{\bar{\alpha}_r\}_{r=0}^k$$

obeys the following properties

$$\begin{aligned} T &< \bar{\alpha}_0 < \bar{\alpha}_1 < \dots < \bar{\alpha}_k, \\ \bar{\alpha}_{r+1} - \bar{\alpha}_r &\sim (1 - c)\pi(T), \quad r = 0, 1, \dots, k-1 \end{aligned}$$

where

$$\pi(T) \sim \frac{T}{\ln T}, \quad T \rightarrow \infty$$

is the prime-counting function and c is the Euler's constant.

7.3. Let us remind that we have proved in our paper [5] the following formula of the Riemann-Siegel type

$$Z'(t) = -2 \sum_{n \leq \rho(t)} \frac{1}{\sqrt{n}} \ln \frac{\rho(t)}{n} \sin\{\vartheta(t) - t \ln n\} + \\ + \mathcal{O}(t^{-1/4} \ln t), \quad \rho(t) = \sqrt{\frac{t}{2\pi}}.$$

Let (comp. [7], (2.1) – (2.3))

$$a_n = \frac{2}{a_n}, \\ \bar{f}_n(t) = \ln \frac{\rho(t)}{n} \sin\{\vartheta(t) - t \ln n\}, \\ \bar{R}(t) = \mathcal{O}(t^{-1/4} \ln t).$$

Then we obtain (comp. [7], (2.5)) from Theorem 4 the following

Corollary 2.

$$(7.5) \quad \prod_{r=1}^k \left| \sum_{n \leq \rho(\bar{\alpha}_r)} a_n f_n(\bar{\alpha}_r) + R(\bar{\alpha}_r) \right| \sim \\ \sim \frac{\Lambda_1}{\sqrt{\left| \sum_{n \leq \rho(\bar{\alpha}_0)} a_n \bar{f}_n(\bar{\alpha}_0) + \bar{R}(\bar{\alpha}_0) \right|}},$$

i.e. an infinite set of metamorphoses of the main multiform (comp. [7], (2.4))

$$G(x_1, \dots, x_k) = \prod_{r=1}^k |Z(x_r)|, \quad \bar{x}_r > T > 0, \quad k \geq 2$$

into quite distinct monoform on the right-hand side of (7.5) corresponds to the infinite subset of the points

$$\{\bar{\alpha}_1(T), \bar{\alpha}_2(T), \dots, \bar{\alpha}_k(T)\}, \quad T \in (T_0, +\infty),$$

where T_0 is sufficiently big.

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